# ARITHMETIC OPERATIONS ON Z-NUMBERS DESCRIBED BY PAIR OF TRAPEZOIDAL MEMBERSHIP FUNCTIONS

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## ABSTRACT

This paper considers implementation of arithmetic operations on Z-numbers described by pairs of trapezoidal membership functions. Trapezoidal numbers allow good approximation for a wide range of fuzzy numbers and are quite frequently used in fuzzy systems. Trapezoidal representation of fuzzy numbers is quite effective and efficient in preserving the systems from the loss of information after series of operations combining fuzzy numbers. We suggest an operational approach to implement arithmetic operations on Z-numbers the components of which are trapezoidal fuzzy numbers.

## 1. INTRODUCTION

### Trapezoidal Fuzzy Model in Representing Z-Numbers

Z-numbers are suggested by Professor Lotfi Zadeh as reliable data structures to store the information at presence of various forms of uncertainty [1,2]. A variable X described by a Z-number has two components (A, B), the first of which specifies the constraints on the values of the variable and the second one is the measure of reliability of the first component. Both components are usually fuzzy numbers.

There are very few researches on practical computation with Z-numbers. The paper [3] considers operations on discrete fuzzy numbers. In our research we consider Z-numbers $Z(A,B)$ where A and B are fuzzy numbers with trapezoidal membership functions.

The following trapezoidal model is suitable for representing botha trapezoidal membership function and a distribution function of probability measure $(a \le b \le c \le d)$:

$$\mu(x) \equiv T((a,b,c,d,h),x) = \begin{cases} 0, & x \le a, \\ h\dfrac{x-a}{b-a}, & a < x < b, \\ h, & b \le x \le c, \\ h\dfrac{d-x}{d-c}, & c < x < d, \\ 0, & x \ge d. \end{cases}$$

When representing a normal MF, we have (h =1):

$$\mu(x) \equiv T\big((a,b,c,d,1),x\big) = \begin{cases} 0, & x \le a, \\ \dfrac{x-a}{b-a}, & a < x < b, \\ 1, & b \le x \le c, \\ \dfrac{d-x}{d-c}, & c < x < d, \\ 0, & x \ge d. \end{cases}$$

For a distribution function of probability measure of random variable $(a < d,\ a \le b \le c \le d)$:

$$P(x) \equiv T\left(\left(a,b,c,d,\frac{2}{(c+d)-(a+b)}\right),x\right) = \begin{cases} 0, & x \le a, \\ \dfrac{2}{(c+d)-(a+b)}\dfrac{x-a}{b-a}, & a < x < b, \\ \dfrac{2}{(c+d)-(a+b)}, & b \le x \le c, \\ \dfrac{2}{(c+d)-(a+b)}\dfrac{d-x}{d-c}, & c < x < d, \\ 0, & x \ge d. \end{cases}$$

Trapezoidal functions can also be described by other parameter sets, when more convenient:

$$\mu_A = T(a_A, b_A, c_A, d_A, 1) \equiv T'(a_A, \Delta_{A1}, \Delta_{A2}, \Delta_{A3}, 1),$$

$$\mu_B = T(a_B, b_B, c_B, d_B, 1) \equiv T'(a_B, \Delta_{B1}, \Delta_{B2}, \Delta_{B3}, 1),$$

$$P(x) \equiv T\left(a_p, b_p, c_p, d_p, \frac{2}{(c_p+d_p)-(a_p+b_p)}\right) \equiv T'\left(a_p, \Delta_{p1}, \Delta_{p2}, \Delta_{p3}, \frac{2}{(\Delta_{p1}+2\Delta_{p2}+\Delta_{p3})}\right),$$

where:

$$\Delta_{A1} = |b_A - a_A|,$$

$$\Delta_{A2} = |c_A - b_A|,$$

$$\Delta_{A3} = |d_A - c_A|,$$

$$\Delta_{B1} = |b_B - a_B|,$$

$$\Delta_{B2} = |c_B - b_B|,$$

$$\Delta_{B3} = |d_B - c_B|,$$

$$\Delta_{p1} = |b_p - a_p|,$$

$$\Delta_{p2} = |c_p - b_p|,$$

$$\Delta_{p3} = |d_p - c_p|,$$

## 2. IMPLEMENTING AN ARITHMETIC OPERATION ON Z-NUMBERS

$$Z = X * Y$$

$$Z(A_Z, B_Z) = Z(A_X, B_X) * Z(A_Y, B_Y).$$

The computation of the variable part ( A ) is quite easy:

$$A_Z = A_X * A_Y$$

where * is any of +, −, ×, /.

For trapezoidal fuzzy numbers (approximation for "×" and " / "):

$$A_X = T(a_X, b_X, c_X, d_X),$$

$$A_Y = T(a_Y, b_Y, c_Y, d_Y),$$

$$A_Z = T(a_Z, b_Z, c_Z, d_Z),$$

$$A_X + A_Y = T(a_X + a_Y, b_X + b_Y, c_X + c_Y, d_X + d_Y),$$

$$A_X - A_Y = T(a_X - d_Y, b_X - c_Y, c_X - b_Y, d_X - a_Y),$$

$$\begin{aligned} A_X \times A_Y = T(&\min(a_X \times a_Y, a_X \times d_Y, d_X \times a_Y, d_X \times d_Y),\\ &\min(b_X \times b_Y, b_X \times c_Y, c_X \times b_Y, c_X \times c_Y),\\ &\max(b_X \times b_Y, b_X \times c_Y, c_X \times b_Y, c_X \times c_Y),\\ &\max(a_X \times a_Y, a_X \times d_Y, d_X \times a_Y, d_X \times d_Y)),\end{aligned}$$

$$A_X / A_Y = T(a_X, b_X, c_X, d_X) \times T(1/d_Y, 1/c_Y, 1/b_Y, 1/a_Y),\ \ a_Y, b_Y, c_Y, d_Y \neq 0.$$

Let's consider the computation of the second part ( B ).

If the probability density functions where known, the resultant probability functions for arithmetic operations could have been computed using the convolutions [4]:

For summation ( + ): $P_Z(v) = \int_R P_X(v-u)P_Y(u)du.$

For subtraction ( − ): $P_Z(v) = \int_R P_X(v+u)P_Y(u)du.$

For multiplication (×): $P_Z(v) = \int_R \frac{1}{|u|} P_X(v/u)P_Y(u)du.$ .4

For division ( / ): $P_Z(v) = \int_R |u| P_X(vu)P_Y(u)du.$

Assume the distributions X p and Y p are trapezoidal functions. Then, $p_{Z=X+Y}(v)$ can be replaced by the parameterized function, which can be computed numerically:

$$p_Z\left(\left(a_{p_X}, b_{p_X}, c_{p_X}, d_{p_X}, a_{p_Y}, b_{p_Y}, c_{p_Y}, d_{p_Y}\right), v\right) =$$

$$Integrate\left(\text{wrt } u \text{ in } [range] \text{ with acc. } \varepsilon,\ T\left(\left(a_{p_X}, b_{p_X}, c_{p_X}, d_{p_X}, h_{p_X}\right), v-u\right) \cdot T\left(\left(a_{p_Y}, b_{p_Y}, c_{p_Y}, d_{p_Y}, h_{p_Y}\right), u\right)\right)$$

Then:

$$\mu_{p_Z}(p_Z) = \sup_{p_X, p_Y}\left(\mu_{B_X}\left(\int_P \mu_{A_\Xi}(u)p_X(u)du\right) \wedge \mu_{B_Y}\left(\int_P \mu_{A_\Psi}(u)p_Y(u)du\right)\right),$$

where pZ is the function defined above, subject to compatibility constraints (COG below stands for "Center Of Gravity"):

$$COG_{\mu_{A_X}} = COG_{p_X}$$

$$COG_{\mu_{A_Y}} = COG_{p_Y}$$

Centroid for a trapezoidal function can be computed as [5]:

$$COG_{T(a,b,c,d,h),x)} = \frac{1}{3}\left(\frac{(c^2+cd+d^2)-(a^2+ab+b^2)}{(c+d)-(a+b)}\right).$$

Hence the compatibility constraint:

$$COG_{T(a_A,b_A,c_A,d_A,1)} = COG_{T'\left(a_p,\Delta_{p1},\Delta_{p2},\Delta_{p3},\frac{2}{\Delta_{p1}+2\Delta_{p2}+\Delta_{p3}}\right)}.$$

Then:

$a_p = S(a_A, b_A, c_A, d_A, \Delta_{p1}, \Delta_{p2}, \Delta_{p3})$, where S(...) is the 7-argument function found from solving the above equation, is a dependent variable.

$$a_p = S(...) = \frac{1}{3}\left(COG_{T(a_A,b_A,c_A,d_A,1)} - \frac{2\Delta_{p1}^2+3\Delta_{p2}^2+\Delta_{p3}^2+6\Delta_{p1}\Delta_{p2}+3\Delta_{p1}\Delta_{p3}+3\Delta_{p2}\Delta_{p3}}{\Delta_{p1}+2\Delta_{p2}+\Delta_{p3}}\right).$$

For our case, the above formula for $\mu(p_Z)$ can be rewritten as:

$$\mu_{p_Z}\left(\left(\Delta^*_{p_{X1}}, \Delta^*_{p_{X2}}, \Delta^*_{p_{X3}}, \Delta^*_{p_{Y1}}, \Delta^*_{p_{Y2}}, \Delta^*_{p_{Y3}}\right), v\right) = \max\Bigg\{$$

$$Integrate\left(wrt\ u, T\left(\left(a_{A_X}, b_{A_X}, c_{A_X}, d_{A_X}, 1\right), u\right) \cdot T'\left(\left(a_{p_X}, \Delta_{p_{X1}}, \Delta_{p_{X2}}, \Delta_{p_{X3}}, \frac{2}{\Delta_{p_{X1}}+2\Delta_{p_{X2}}+\Delta_{p_{X3}}}\right), u\right)\right),$$

$$Integrate\left(wrt\ u, T\left(\left(a_{A_Y}, b_{A_Y}, c_{A_Y}, d_{A_Y}, 1\right), u\right) \cdot T'\left(\left(a_{p_Y}, \Delta_{p_{Y1}}, \Delta_{p_{Y2}}, \Delta_{p_{Y3}}, \frac{2}{\Delta_{p_{Y1}}+2\Delta_{p_{Y2}}+\Delta_{p_{Y3}}}\right), u\right)\right)$$

$$\Bigg\} \rightarrow \max_{\substack{\Delta_{p_{X1}}, \Delta_{p_{X2}}, \Delta_{p_{X3}}, \\ \Delta_{p_{Y1}}, \Delta_{p_{Y2}}, \Delta_{p_{Y3}}}},$$

subject to:

For $Z = X + Y$ (summation):

$$\mu_{p_Z}\left(\left(\Delta^*_{p_{X1}}, \Delta^*_{p_{X2}}, \Delta^*_{p_{X3}}, \Delta^*_{p_{Y1}}, \Delta^*_{p_{Y2}}, \Delta^*_{p_{Y3}}\right), v\right) =$$

$$Integrate\left(wrt\ u, T'\left(\left(a_{p_X}, \Delta^*_{p_{X1}}, \Delta^*_{p_{X2}}, \Delta^*_{p_{X3}}, \Delta^*_{p_{Y1}}, h_{p_X}\right), v-u\right) T\left(\left(a_{p_Y}, \Delta^*_{p_{Y1}}, \Delta^*_{p_{Y2}}, \Delta^*_{p_{Y3}}, h_{p_Y}\right), u\right)\right)$$

For $Z = X - Y$ (subtraction):

$$\mu_{p_Z}\left(\left(\Delta^*_{p_{X1}}, \Delta^*_{p_{X2}}, \Delta^*_{p_{X3}}, \Delta^*_{p_{Y1}}, \Delta^*_{p_{Y2}}, \Delta^*_{p_{Y3}}\right), v\right) =$$

$$Integrate\left(wrt\ u, T'\left(\left(a_{p_X}, \Delta^*_{p_{X1}}, \Delta^*_{p_{X2}}, \Delta^*_{p_{X3}}, \Delta^*_{p_{Y1}}, h_{p_X}\right), v+u\right)\cdot T\left(\left(a_{p_Y}, \Delta^*_{p_{Y1}}, \Delta^*_{p_{Y2}}, \Delta^*_{p_{Y3}}, h_{p_Y}\right), u\right)\right).$$

For $Z = X \times Y$ (multiplication):

$$\mu_{p_Z}\left(\left(\Delta^*_{p_{X1}}, \Delta^*_{p_{X2}}, \Delta^*_{p_{X3}}, \Delta^*_{p_{Y1}}, \Delta^*_{p_{Y2}}, \Delta^*_{p_{Y3}}\right), v\right) =$$

$$Integrate\left(wrt\ u, \frac{1}{|u|}T'\left(\left(a_{p_X}, \Delta^*_{p_{X1}}, \Delta^*_{p_{X2}}, \Delta^*_{p_{X3}}, \Delta^*_{p_{Y1}}, h_{p_X}\right), v/u\right)\cdot T\left(\left(a_{p_Y}, \Delta^*_{p_{Y1}}, \Delta^*_{p_{Y2}}, \Delta^*_{p_{Y3}}, h_{p_Y}\right), u\right)\right).$$

For $Z = X / Y$ (division):

$$\mu_{p_Z}\left(\left(\Delta^*_{p_{X1}}, \Delta^*_{p_{X2}}, \Delta^*_{p_{X3}}, \Delta^*_{p_{Y1}}, \Delta^*_{p_{Y2}}, \Delta^*_{p_{Y3}}\right), v\right) =$$

$$Integrate\left(wrt\ u, |u| T'\left(\left(a_{p_X}, \Delta^*_{p_{X1}}, \Delta^*_{p_{X2}}, \Delta^*_{p_{X3}}, \Delta^*_{p_{Y1}}, h_{p_X}\right), vu\right)\cdot T\left(\left(a_{p_Y}, \Delta^*_{p_{Y1}}, \Delta^*_{p_{Y2}}, \Delta^*_{p_{Y3}}, h_{p_Y}\right), u\right)\right).$$

Then:

$$a_{p_X} = S(a_{A_X}, b_{A_X}, c_{A_X}, d_{A_X}, \Delta_{p_{X1}}, \Delta_{p_{X2}}, \Delta_{p_{X3}}),$$

$$a_{p_X} = S(a_{A_Y}, b_{A_Y}, c_{A_Y}, d_{A_Y}, \Delta_{p_{Y1}}, \Delta_{p_{Y2}}, \Delta_{p_{Y3}}).$$

subject to:

$$w = \int_P \mu_{A_Z}(u) p_Z(u) du.$$

The computation of $\mu_{B_Z}(w)$ for any given w in our case can be done using the algorithm described below.

For a value w find such values of parameters $\left(a_{p_X}, b_{p_X}, c_{p_X}, d_{p_X}, a_{p_Y}, b_{p_Y}, c_{p_Y}, d_{p_Y}\right)$ that minimize the expression:

$$\left[Integrate\left(wrt\ u, T\left(\left(a_{A_Z}, b_{A_Z}, c_{A_Z}, d_{A_Z}, 1\right), u\right)\cdot p_Z\left(a_{p_X}, b_{p_X}, c_{p_X}, d_{p_X}, a_{p_Y}, b_{p_Y}, c_{p_Y}, d_{p_Y}\right), u\right) - w\right]^2 \to$$

$$\to \min_{\substack{a_{p_X}, b_{p_X}, c_{p_X}, d_{p_X}, \\ a_{p_Y}, b_{p_Y}, c_{p_Y}, d_{p_Y}}}$$ and

maximize the expression:

$$\mu_{p_Z} = \left(b_{p_X} - a_{p_X}, c_{p_X} - b_{p_X}, d_{p_X} - c_{p_X}, b_{p_Y} - a_{p_Y}, c_{p_Y} - b_{p_Y}, d_{p_Y} - c_{p_Y}\right) \to \min_{\substack{a_{p_X}, b_{p_X}, c_{p_X}, d_{p_X}, \\ a_{p_Y}, b_{p_Y}, c_{p_Y}, d_{p_Y}}}$$

The corresponding value for ( ) Z B w μ then would be:

$$\mu_{B_Z}(w) = \mu_{p_Z}\left(b_{p_X} - a_{p_X}, c_{p_X} - b_{p_X}, d_{p_X} - c_{p_X}, b_{p_Y} - a_{p_Y}, c_{p_Y} - b_{p_Y}, d_{p_Y} - c_{p_Y}\right).$$

## 3. NOTE ON PROGRAM IMPLEMENTATION

All of the algorithms described above are implemented using an evolutionary optimization method. We use the DE algorithm [6,7] with constraints as one of the best one for numerical optimization.

Substitution of variables to ensure account of constraints for trapezoids (e.g. $a \le b \le c \le d$) and effective DE optimization (almost the same domain of variation for all optimization variables) is done as follows:

Optimization variables: $t_i$, $i = 1,\ldots,8$.

$$a_{p_X} = t_1$$

$$b_{p_X} = t_1 + |t_2 - t_1|$$
$$c_{p_X} = t_1 + |t_2 - t_1| + |t_3 - t_2|$$
$$d_{p_X} = t_1 + |t_2 - t_1| + |t_3 - t_2| + |t_4 - t_3|$$
$$a_{p_Y} = t_5$$
$$b_{p_Y} = t_5 + |t_6 - t_5|$$
$$c_{p_Y} = t_5 + |t_6 - t_5| + |t_7 - t_6|$$
$$d_{p_Y} = t_5 + |t_6 - t_5| + |t_7 - t_6| + |t_8 - t_7|$$

## 4. CONCLUSION

We presented a set of algorithms that can be used to implement of arithmetic operations over Z-numbers. All underlying optimization is to be done by one of leading evolutionary optimization method, a version of DE with constraints, implemented by the author.

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